5. POTETYUNKO E.N. and SRUBSHCHIK L.S., Asymptotic analysis of the wave motions of a viscous fluid with a free boundary, Prikl. Matem. i Mekhan., 34, 5, 1970.
6. MIROPOL'SKII YU.Z., On the generation of internal waves in the ocean by a wind field, Okeanologiya, 10, 3, 1975.
7. BATYSHCHEV V.A., Asymptotic forms of the non-uniform warming of the free boundary of a capillary fluid at high Marangoni numbers, Prikl. Matem. i Mekhan., 53, 3, 1989.
8. BATYSHCHEV V.A., Asymptotic forms of axially symmetric flows of a fluid with a free boundary in the case of a disappearing viscosity, Prikl. Matem. i Tekhn. Fiz., 3, 1975.
9. VISHIK M.A. and LYUSTERNIK L.A., Regular confluence and the boundary layer for linear differential equations with a small parameter, Uspekhi Mat. Nauk, 12, 5, 1957.
10. OVSYANNIKOV L.V., MAKARENKO N.I., NALIMOV V.I., et al., Non-Linear Problems in the Theory of Surface and Internal Waves, Nauka, Novosibirsk, 1985.

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THE GENERALIZED PROBLEM OF BREAKUP OF AN ARBITRARY DISCONTINUITY*

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The problem of breakup of an arbitrary discontinuity in a gas (the Riemann problem) is generalized to the case when an arbitrary, in general space-variable, distribution of the gas-dynamic parameters is given on both sides of the discontinuity at the initial instant of time (the generalized Riemann problem /1/). The solvability of this, in general non-selfsimilar, model is proved and analytical formulas are found for its solution in a small neighbourhood of the points of discontinuity in the x, t plane, where x is the space coordinate and t is the time.

A complete analysis of the selfsimilar Riemann problem was developed by Kochin /2/. The generalized Riemann problem is in general non-selfsimilar and does not admit of a simple analytical solution over the entire x, t plane. However, some analytical solutions may be obtained for this problem. Thus, for a linear initial distribution, analytical formulas were obtained in /1/ for the values of the derivatives of the gas-dynamic parameters along the contact discontinuity for $t=0$.

Below, the generalized Riemann problem is considered in a small neighbourhood of the point of discontinuity in the (x, t) plane and its analytical solution is constructed to a first approximation in $\theta=\sqrt{x^2+t^2}$. Analytical formulas for the trajectories of discontinuities are obtained in the same approximation.

1. The generalized Riemann problem is reducible to the following Cauchy problem for one-dimensional non-stationary equations of gas dynamics:

$$(\rho\varphi)_t+(\rho u\varphi+\mathbf{F})_x=0 \tag{1.1}$$
$$\varphi=(1,\ u,\ e+{}^1/_2u^2)^T,$$
$$\mathbf{F}=(0,p,pu)^T,\quad \rho\varphi(0,x)=\begin{cases}\varphi_1(x), & x<0\\ \varphi_2(x), & x>0\end{cases}$$

where u, ρ and e are the velocity, density, and the specific internal energy, $p=p(\rho,\ e)$ is the pressure and $\varphi_1(x)$ and $\varphi_2(x)$ are functions which are differentiable in the domain of definition, which specify the initial parameter distribution.

We will rewrite the system of Eqs.(1.1) in characteristic form, introducing the specific

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entropy $s = s(\rho, e)$, the velocity of sound $a = a(\rho, e)$, the mass velocity of sound $c = \rho a$, and also new variables θ and λ related to x and t by $\theta = \sqrt{x^2 + t^2}$ and $\lambda = xt^{-1}$:

$$cD_+u + D_+p = 0,\ cD_-u - D_-p = 0,\ Ds = 0 \tag{1.2}$$
$$D = \frac{\sqrt{1+\lambda^2}}{\theta}(u - \lambda)\left[\frac{\partial}{\partial\lambda} + \left(\frac{\lambda}{1+\lambda^2} + \frac{1}{u-\lambda}\right)\theta\frac{\partial}{\partial\theta}\right]$$
$$D_\pm = \frac{\sqrt{1+\lambda^2}}{\theta}(u \pm a - \lambda)\left[\frac{\partial}{\partial\lambda} + \left(\frac{\lambda}{1+\lambda^2} + \frac{1}{u \pm a - \lambda}\right)\theta\frac{\partial}{\partial\theta}\right]$$

The solution of the system of Eqs.(1.2) to within $O(\theta^2)$ is sought in the form

$$u = u_0(\lambda) + \theta u_1(\lambda),\ p = p_0(\lambda) + \theta p_1(\lambda) \tag{1.3}$$
$$s = s_0(\lambda) + \theta s_1(\lambda)$$

Considering the zeroth approximation with respect to θ, we find that the function with subscript zero in (1.3) may be of one of the following three types: either $u_0, p_0, s_0 = \text{const}$, which corresponds to a homogeneous constant flow, or it may satisfy the relationships

$$u_0 \mp a_0 = \lambda,\ c_0u_0' \pm p_0' = 0,\ s_0 = \text{const} \tag{1.4}$$

which corresponds to a rarefaction wave (the upper sign is for a wave propagating to the left and the lower sign is for a wave propagating to the right); the prime denotes derivatives with respect to λ.

Given the zeroth-approximation solution, we will consider the first approximation in θ for each of the three types of solutions. Unless otherwise stated, we denote the first terms (the zeroth approximation) by the subscript 0 and the coefficients of θ (the first approximation) by the subscript 1 in expansions of the form (1.3).

Constant flow. Integrating the corresponding system of equations in the first approximation

$$c_0D_+^*u_1 + D_+^*p_1 = 0,\ c_0D_-^*u_1 - D_-^*p_1 = 0,\ D^*s = 0 \tag{1.5}$$
$$D_\pm^* = \frac{d}{d\lambda} + \left(\frac{\lambda}{1+\lambda^2} + \frac{1}{u_0 \pm a_0 - \lambda}\right),\quad D^* = \frac{d}{d\lambda} + \left(\frac{\lambda}{1+\lambda^2} + \frac{1}{u_0 - \lambda}\right)$$

we obtain the general solution

$$\begin{Bmatrix} u_1 \\ p_1/c_0 \end{Bmatrix} = C_1\frac{u_0 + a_0 - \lambda}{2\sqrt{1+\lambda^2}} \pm C_2\frac{u_0 - a_0 - \lambda}{2\sqrt{1+\lambda^2}} \tag{1.6}$$
$$s_1 = C_3\frac{u_0 - \lambda}{\sqrt{1+\lambda^2}}$$

where C_1, C_2, C_3 are arbitrary constants.

Rarefaction wave. To be specific, we will consider the case when the wave propagates to the right. Then the system of equations in the first approximation takes the form

$$c_0u_1 + p_1 + (u_1 + a_1)(c_0u_0' + p_0') = 0,\ D^*s_1 = 0 \tag{1.7}$$
$$D_-^*u_1 - \frac{1}{c_0}D_-^*p_1 = \left[\left(\frac{\partial c^{-1}}{\partial p}\right)_0 p_1 + \left(\frac{\partial c^{-1}}{\partial s}\right)_0 s_1\right]p_0'$$

The operators $D_\pm^*$ and D^* have the same form as in (1.5), and the variables with subscript zero (in particular, those in the operators D^* and $D_\pm^*$) are functions of λ which satisfy relationships (1.4) with the lower sign.

Integration of the second equation in (1.7) gives

$$s_1 = C_2(1+\lambda^2)^{-1/2}\exp\left[-\int\frac{d\lambda}{u_0 - \lambda}\right] \tag{1.8}$$

where C_2 is an arbitrary constant. Seeing that

$$a_1 = \left(\frac{\partial a}{\partial p}\right)_0 p_1 + \left(\frac{\partial a}{\partial s}\right)_0 s_1$$

we can express u_1 from the first equation in (1.7)

$$u_1 = K_0(\lambda)\, p_1 + L_0(\lambda)\, s_1 \qquad (1.9)$$
$$K_0(\lambda) = -\frac{1}{c_0}\,\frac{1+2a_0'}{1+2u_0'}, \quad L_0(\lambda) = -\frac{2u_0'}{1+2u_0'}\left(\frac{\partial a}{\partial s}\right)_0$$

Using this relationship to eliminate u_1 from the second equation in (1.7), we obtain an equation for

$$D_-^*\,[(K_0 - c_0^{-1})\, p_1 + L_0 s_1] = (\partial c^{-1}/\partial s)_0\, p_0' s_1$$

whose general solution is

$$p_1 = C\left(K_0 - \frac{1}{c_0}\right)^{-1}(1+\lambda^2)^{-1/2}\exp\left[-\int\frac{d\lambda}{u_0 - a_0 - \lambda}\right] \qquad (1.10)$$
$$C = C(\lambda) = \int\sqrt{1+\lambda^2}\,\exp\left[\frac{d\lambda}{u_0 - a_0 - \lambda}\right]\left[\left(\frac{\partial c^{-1}}{\partial s}\right)_0 p_0' s_1 - D_-^*(L_0 s_1)\right] d\lambda + C_1$$

where C_1 is an arbitrary constant.

Formulas (1.8)-(1.10) give a general solution in the first approximation for the case of a right-propagating rarefaction wave in the zeroth approximation. We can similarly determine the solution for the left-propagating rarefaction wave. These solutions are essentially simplified if we use relationships (1.4) for the zeroth approximation. Indeed, for the centred rarefaction wave we have the equality

$$\exp\left[-\int\frac{d\lambda}{u_0 - \lambda}\right] = c_0$$

Using this equality in combination with (1.4), we transform formulas (1.8)-(1.10) (and the corresponding formulas for the left wave) and obtain the general solution in the first approximation for the rarefaction wave in the form

$$p_1 = \frac{c_0\sqrt{c_0}}{(c_0 K_0 \pm 1)\sqrt{1+\lambda^2}}\,[C_1 + C_2\Omega_0] \qquad (1.11)$$
$$s_1 = C_2\,(1+\lambda^2)^{-1/2} c_0, \quad u_1 = K_0 p_1 + L_0 s_1$$
$$K_0 = \pm\frac{3 - 2u_0'}{c_0(1+2u_0')}, \quad L_0 = \pm\frac{2u_0'}{1+2u_0'}\left(\frac{\partial a}{\partial s}\right)_0$$
$$\Omega_0 = -\int\sqrt{c_0}\left[\left(\frac{\partial \ln c}{\partial s}\right)_0 u_0' + L_0' + \frac{L_0}{2(\lambda - u_0)}\right] d\lambda$$

where the upper sign corresponds to a left-propagating rarefaction wave and the lower sign to the right-propagating wave; the parameters with subscript 0 are defined by (1.4); C_1 and C_2 are arbitrary constants.

We have thus obtained an analytical expression for the general solution in the first approximation both in the constant flow region and in the rarefaction wave region. Note that, in the first case, the general solution is determined apart from three arbitrary constants and in the second case apart from two constants.

When solving the classical Riemann problem, the entire flow region is divided into a number of subregions separated from one another by parameter discontinuity surfaces. Each subregion is characterized either by constant flow or by a rarefaction wave. The general solution in these subregions is known: it is described either by (1.6) or by (1.8)-(1.11). Therefore, the solution of the generalized Riemann problem has been constructed, but only apart from a number of arbitrary constants ($\leqslant 16$, depending on the particular problem). The values of these constants should be determined by analysing the relationships on the parameter discontinuity surfaces. In the next section we will show that in the first approximation a complete analytical solution is obtained for the generalized Riemann problem, i.e., we determine all the constants and also the discontinuity trajectories in the x, t plane.

2. In the zeroth approximation with respect to θ (the classical case), the discontinuity breaks up into a contact surface with right- and left-adjacent constant flow regions. These constant-flow regions are separated from unperturbed regions by a shock wave, or by a rarefaction wave fan, or finally by a weak discontinuity. We assume that the wave picture is determined by the classical case, i.e., if initially the decay of the discontinuity produces a shock wave, say, then the shock wave persists over some finite time interval.

Consider the region lying on one side of the contact discontinuity (on the left or on the right). We will show that in this region the solution of the generalized Riemann problem is determined apart from one arbitrary constant.

To fix our ideas, let us consider the left region. Introduce a double index for the parameters of the medium: the second index identifies the order of the approximation and the first index identifies the subregion where the solution is sought (1 for the unperturbed region, 2 for the rarefaction wave and 3 for the constant flow adjacent to the contact surface: see Fig.1). Thus, for instance, the pressure in the rarefaction wave is written in the form $p_2 = p_{20}(\lambda) + \theta p_{21}(\lambda) + O(\theta^2)$. The discontinuity surfaces are also indexed: 1 and 2 are the characteristics bounding the rarefaction wave and 3 is the contact surface (Fig.1). In the case of a shock wave or a weak discontinuity, region 2 disappears and trajectories 1 and 2 merge into one.

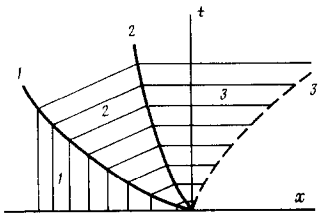

Fig.1

The equations of the trajectories of these discontinuities are represented in the form $\lambda_i = \lambda_{i0} + \lambda_{i1}\theta + O(\theta^2)$, $i = 1, 2, 3$. Then their velocities are $D_i = \lambda_{i0} + 2\lambda_{i1}\theta + O(\theta^2)$.

We assume that the solution of the problem in the zeroth approximation (the parameters with subscript 0) is known /3/. Arbitrary constants in general solutions for regions 1, 2, 3 will be determined from "matching" conditions on the discontinuity surfaces.

Assume that the initial parameter distribution on the left and on the right of the discontinuity is given in its neighbourhood in the form of a series in powers of θ:

$$\lambda = -\infty, \quad u = u_{10} + U_{10}\theta + O(\theta^2)$$
$$p = p_{10} + P_{10}\theta + O(\theta^2), \quad s = s_{10} + S_{10}\theta + O(\theta^2) \tag{2.1}$$

The general solution in region 1 is given by formulas (1.6). Using the condition for this solution to be "matched" with the initial values (2.1), specifically

$$\lim_{\lambda\to-\infty} u_{11}(\lambda) = U_{10}, \ \lim_{\lambda\to-\infty} p_{11}(\lambda) = P_{10}, \ \lim_{\lambda\to-\infty} s_{11}(\lambda) = S_{10}$$

we can determine the arbitrary constants and write the solution in region 1 in the form

$$u_{11} = (1 + \lambda^2)^{-1/2}\left[(u_{10} - \lambda)\, U_{10} + P_{10}/\rho_{10}\right]$$
$$p_{11} = \frac{(u_{10} - \lambda)\, P_{10} + c_{10}a_{10}U_{10}}{\sqrt{1+\lambda^2}}, \quad s_{11} = \frac{u_{10} - \lambda}{\sqrt{1+\lambda^2}} S_{10} \tag{2.2}$$

If relationships (2.1) are a representation of the initial values on the right of the contact discontinuity $(x > 0)$, then by matching the general solution (1.6) with the initial values on the line $\lambda = +\infty$ we similarly determine the first-approximation solution for the unperturbed region on the right:

$$u_{11} = \frac{(\lambda - u_{10})\, U_{10} - P_{10}/\rho_{10}}{\sqrt{1+\lambda^2}}$$
$$p_{11} = \frac{(\lambda - u_{10})\, P_{10} - c_{10}a_{10}U_{10}}{\sqrt{1+\lambda^2}}, \quad s_{11} = \frac{\lambda - u_{10}}{\sqrt{1+\lambda^2}} S_{10}$$

The solution in the unperturbed region is thus completely determined.

Now consider the "matching" of this solution with the solution in region 3. We have to consider three cases, when these regions are separated by a shock wave, a weak discontinuity, and a rarefaction wave.

Shock wave. On the shock wave $\lambda = \lambda_1(\theta)$ we have the Rankine-Hugoniot conditions

$$\rho_3 v_3 = m_1, \ p_3 + m_1 v_3 = I_1, \ {}^1/_2 v_3^2 + h_3 = H_1 \tag{2.3}$$

where $v_3 = D_1 - u_3$, h_3 is the enthalpy and m_1, I_1 and H_1 are the mass, momentum, and energy fluxes, determined from the parameter values ahead of the wave (in region 1).

Substituting expansions of the form (1.4) into (2.3) and considering the first approximation in θ, we obtain a system of linear equations for the parameter values in region 3 on the shock wave:

$$\lambda = \lambda_{10}, \quad \rho_{30}v_{31} + v_{30}\left[\left(\frac{\partial\rho}{\partial p}\right)_{30} p_{31} + \left(\frac{\partial\rho}{\partial s}\right)_{30} s_{31}\right] = m_{11} \tag{2.4}$$
$$p_{31} + m_{10}v_{31} + m_{11}v_{30} = I_{11}$$
$$v_{30}v_{31} + \left(\frac{\partial h}{\partial p}\right)_{30} p_{31} + \left(\frac{\partial h}{\partial s}\right)_{30} s_{31} = H_{11}$$

Omitting the fairly complex procedure for solving this system, we will merely present the final result: the values of the first-approximation parameters in region 3 on the shock wave front $(\lambda = \lambda_{10})$ are represented in the form

$$\begin{pmatrix} u_{31} \\ p_{31} \\ s_{31} \end{pmatrix} = \begin{pmatrix} U_1 \\ P_1 \\ S_1 \end{pmatrix} + 2\lambda_{11} \begin{pmatrix} U_2 \\ P_2 \\ S_2 \end{pmatrix} \tag{2.5}$$

$$S_1 = \left(\frac{\partial h}{\partial s}\right)_{30}^{-1} \left[\left(1 - \frac{\rho_{10}}{\rho_{30}}\right)(v_{30} - v_{10}) u_{11} + \left(\frac{1}{\rho_{10}} - \frac{1}{\rho_{30}}\right)\left(1 - \frac{\rho_{10}}{\rho_{30}} M_{10}{}^2\right) p_{11} + \left(\frac{\partial h}{\partial s}\right)_{10} s_{11} + \frac{v_{10}(v_{30} - v_{10})}{\rho_{30}} \left(\frac{\partial \rho}{\partial s}\right)_{10} s_{11}\right], \quad U_1 = -\frac{1}{\rho_{30}(1 - M_{30}{}^2)} \left\{\left[2\rho_{30} M_{30}{}^2 - \rho_{10}(1 + M_{30}{}^2)\right] u_{11} - v_{30} \left(\frac{\partial \rho}{\partial s}\right)_{30} S_1 + \left(1 + M_{30}{}^2 - \frac{\rho_{30}}{\rho_{10}} M_{30}{}^2\right) v_{10} \left(\frac{\partial \rho}{\partial s}\right)_{10} s_{11} + \frac{\rho_{10} M_{10}{}^2 (1 + M_{30}{}^2) - \rho_{30} M_{30}{}^2 (1 + M_{10}{}^2)}{m_{10}} p_{11}\right\}, \quad P_1 = m_{10} \left[\left(\frac{\rho_{10}}{\rho_{30}} - 2\right) u_{11} + U_1\right] + \left(1 + M_{10}{}^2 - \frac{\rho_{10}}{\rho_{30}} M_{10}{}^2\right) p_{11} - v_{10}(v_{30} - v_{10}) \left(\frac{\partial \rho}{\partial s}\right)_{10} s_{11}$$

$$S_2 = \left(\frac{\partial h}{\partial s}\right)_{30}^{-1} \left(\frac{\rho_{10}}{\rho_{30}} - 1\right)(v_{30} - v_{10}), \quad U_2 = 1 - \frac{1}{\rho_{30}(1 - M_{30}{}^2)} \left[\rho_{10}(1 + M_{30}{}^2) - 2\rho_{30} M_{30}{}^2 - v_{30} \left(\frac{\partial \rho}{\partial s}\right)_{30} S_2\right], \quad P_2 = \left(1 - \frac{\rho_{10}}{\rho_{30}} + U_2\right) m_{10} \tag{2.6}$$

$$v_{10} = \lambda_{10} - u_{10}, \; v_{30} = \lambda_{10} - u_{30}, \; M_{10} = v_{10} a_{10}{}^{-1}, \; M_{30} = v_{30} a_{30}{}^{-1}$$

$$m_{10} = \rho_{10} v_{10} = \rho_{30} v_{30}$$

where s_{11}, u_{11} and p_{11} are the values of the first-approximation parameters in the unperturbed zone (2.2) for $\lambda = \lambda_{10}$.

Substituting the parameter values on the shock wave (2.5) in the general solution for region 3 (1.6), we can determine the constants and thus find the solution in the zone adjacent to the contact surface in the shock wave case. Omitting the intermediate steps, we will give the final result: the first approximation in the region between the shock wave and the contact discontinuity has the form

$$u_{31}(\lambda) = U_1(\lambda) + C U_2(\lambda) \tag{2.7}$$

$$p_{31}(\lambda) = P_1(\lambda) + C P_2(\lambda), \; s_{31}(\lambda) = S_1(\lambda) + C S_2(\lambda)$$

$$U_i(\lambda) = U_i \varphi_1(\lambda) + c_{30}{}^{-1} P_i \varphi_2(\lambda) \tag{2.8}$$

$$P_i(\lambda) = c_{30} U_i \varphi_2(\lambda) + P_i \varphi_1(\lambda), \; S_i(\lambda) = S_i \varphi_3(\lambda), \; i = 1, \; 2$$

$$\varphi_1(\lambda) = \sqrt{\frac{1 + \lambda_{10}{}^2}{1 + \lambda^2}} \frac{(u_{30} - \lambda)(u_{30} - \lambda_{10}) - a_{30}{}^2}{(u_{30} - \lambda_{10})^2 - a_{30}{}^2}$$

$$\varphi_2(\lambda) = \sqrt{\frac{1 + \lambda_{10}{}^2}{1 + \lambda^2}} \frac{a_{30}(\lambda - \lambda_{10})}{(u_{30} - \lambda_{10}{}^2) - a_{30}{}^2}$$

$$\varphi_3(\lambda) = \sqrt{\frac{1 + \lambda_{10}{}^2}{1 + \lambda^2}} \frac{u_{30} - \lambda}{u_{30} - \lambda_{10}}$$

The constant C in (2.7) equals $2\lambda_{11}$, i.e., it is equal to the coefficient of θ in the expansion of the shock wave velocity D_1 in powers of θ. This constant is arbitrary. Thus, the first-approximation solution as a whole in the region to the left of the contact discontinuity for the shock-wave case is described by formulas (2.7)-(2.8) and has exactly one arbitrary constant.

Weak discontinuity. In this case the shock wave degenerates into a characteristic on which all the parameters are equal. Using the condition of equality for the first-approximation parameters and noting that the values on the characteristics in the unperturbed zone are known ((2.2) for $\lambda = \lambda_{10}$) and the general solution in the region behind the characteristic is given by formulas (1.6), we can write a system of linear equations for the three arbitrary constants in the general solution. The rank of the matrix of this system is 2, so that we can only find two constants and therefore the solution in region 3 can be obtained apart from one constant. This solution may be represented in the form (2.7), as for the shock wave case, but the corresponding functions are different:

$$U_1(\lambda) = \varphi_1(\lambda) u_{11}(\lambda_{10}), \quad U_2(\lambda) = \frac{u_{30} + \varepsilon a_{30} - \lambda}{2\sqrt{1 + \lambda^2}} \tag{2.9}$$

$$P_1(\lambda) = \varphi_1(\lambda) p_{11}(\lambda_{10}), \; P_2(\lambda) = \varepsilon c_{30} U_2(\lambda)$$

$$S_1(\lambda) = \Lambda_1 \frac{u_{30} - \lambda}{u_{30} - \lambda_{10}} s_{11}(\lambda_{10}), \quad S_2(\lambda) = 0$$

$$\varphi_1(\lambda) = \Lambda_1 \frac{u_{30} - \varepsilon a_{30} - \lambda}{u_{30} - \varepsilon a_{30} - \lambda_{10}}, \quad \Lambda_1 = \sqrt{\frac{1 + \lambda_{10}{}^2}{1 + \lambda^2}}, \quad \varepsilon = -1$$

The curvature of the characteristic in the first approximation is described by the relation

$$\lambda_{11} = 1/2\,[u_{11}(\lambda_{10}) + \varepsilon a_{11}(\lambda_{10})]$$

Rarefaction wave. In this region, the entire set of solutions is defined by two arbitrary constants. "Matching" of two solutions on the first characteristic - the solution in the unperturbed region and the solution in the shock wave region - produces the following conditions in the first approximation:

$$\lambda = \lambda_{10}, \quad u_{21} + \lambda_{11}u_{20}' = u_{11}$$
$$p_{21} + \lambda_{11}p_{20}' = p_{11}, \quad s_{21} = s_{11} \tag{2.10}$$

From these three equations we can find two arbitrary constants C_1, C_2 in formulas (1.11) and the value of λ_{11} characterizing the curvature of the first characteristic of the rarefaction wave. Omitting the intermediate steps, we give the final result: the first approximation in the rarefaction wave is described by the formulas

$$p_{21} = \Lambda_1 (c_{20}K_{20} - \varepsilon)^{-1} (c_{20}/c_{10})^{3/2} [-2\varepsilon p_{11}(\lambda_{10}) -$$
$$c_{10}L_{20}(\lambda_{10})\, s_{11}(\lambda_{10}) + \sqrt{c_{10}}s_{11}(\lambda_{10})\, \Omega_{20}\,|_{\lambda_{10}}^{\lambda}]$$
$$s_{21} = \Lambda_1 (c_{20}/c_{10})\, s_{11}(\lambda_{10}), \quad u_{21} = K_{20}p_{21} + L_{20}s_{21} \tag{2.11}$$

and the curvature of the first characteristic is given by

$$\lambda_{11} = \left.\frac{u_{11} - K_{20}p_{11} - L_{20}s_{11}}{u_{20}'(1 - \varepsilon c_{10}K_{20})}\right|_{\lambda=\lambda_{10}} \tag{2.12}$$

Here all quantities with the subscript 20 are functions of λ satisfying the relationships for the left centred wave (1.5) and $\varepsilon = -1$ as in (2.9).

The solution in region 3 adjacent to the contact discontinuity is described by formulas (1.6) with three arbitrary constants. These constants are determined from the condition for this solution to be "matched" with solution (2.11) on the second characteristic:

$$\lambda = \lambda_{20}, \quad u_{31} = u_{21} + u_{20}'\lambda_{21}$$
$$p_{31} = p_{21} + p_{20}'\lambda_{21}, \quad s_{31} = s_{21} \tag{2.13}$$

Seeing that on the left characteristic $\lambda = \lambda_{20}$, we have $c_{30}u_{31} - p_{31} = 0$, which follows directly from (1.6), we obtain from (2.13) the ratio that characterizes the curvature of the second characteristic in the first approximation:

$$\lambda_{21} = -\frac{\varepsilon p_{21}(\lambda_{20}) + c_{30}u_{21}(\lambda_{20})}{2u_{20}'(\lambda_{20})\, c_{30}}$$

Two relationships remain for three constants. Therefore, for a rarefaction wave, the solution in region 3 has one arbitrary constant. After reduction, this solution can be represented in the form (2.7) with the functions

$$U_1(\lambda) = \frac{1}{2}\sqrt{\frac{1+\lambda_{20}^2}{1+\lambda^2}}\,\frac{u_{30} - \varepsilon a_{30} - \lambda}{u_{30} - \varepsilon a_{30} - \lambda_{20}}\,\frac{c_{30}u_{21}(\lambda_{20}) - \varepsilon p_{21}(\lambda_{20})}{c_{30}}$$
$$U_2(\lambda) = \frac{\lambda_{20} - \lambda}{2\sqrt{1+\lambda^2}}, \quad P_1(\lambda) = -\varepsilon c_{30}U_1(\lambda), \quad P_2(\lambda) = \varepsilon c_{30}U_2(\lambda)$$
$$S_1(\lambda) = \frac{c_{30}}{c_{10}}\sqrt{\frac{1+\lambda_{10}^2}{1+\lambda^2}}\,\frac{u_{30} - \lambda}{u_{30} - \lambda_{20}}\, s_{11}(\lambda_{10}), \quad S_2(\lambda) = 0 \tag{2.14}$$

Thus, the solution in region 3 adjacent to the contact discontinuity on the left side has the form (2.7) and (2.8) if there is a shock wave on the left of the contact discontinuity, (2.9) if there is a weak discontinuity, and (2.14) if there is a rarefaction wave. This solution is determined apart from one arbitrary constant. The same conclusion is reached also if we consider the region on the right of the contact discontinuity. All the formulas obtained in Sect.2 remain true if we take $\varepsilon = 1$.

Denote by C_l and C_r constants that determine the solutions on the left and on the right of the contact discontinuity, respectively. To find these constants, we apply the equality of velocities and pressures on the contact surface, which to a first approximation has the form

$$u_{31}(-\lambda_{30}) = u_{31}(+\lambda_{30}),\ p_{31}(-\lambda_{30}) = p_{31}(+\lambda_{30})$$

Substituting in these relationships the solutions (2.7) in the regions adjacent to the contact discontinuity, we obtain a system of equations for the constants C_l and C_r:

$$U_{1l} + C_l U_{2l} = U_{1r} + C_r U_{2r}, \quad P_{1l} + C_l P_{2l} = P_{1r} + C_r P_{2r}$$

where the subscript l denotes the values of the corresponding functions for $\lambda \to -\lambda_{30}$ and the subscript r denotes those for $\lambda \to +\lambda_{30}$. Solving this system, we obtain

$$C_l = \frac{[U_1] P_{2r} - [P_1] U_{2r}}{U_{2l} P_{2r} - U_{2r} P_{2l}}, \quad C_r = \frac{[U_1] P_{2l} - [P_1] U_{2l}}{U_{2l} P_{2r} - U_{2r} P_{2l}} \qquad (2.15)$$
$$[U_1] = U_{1r} - U_{1l},\ [P_1] = P_{1r} - P_{1l}$$

which thus determines the complete solution of the generalized Riemann problem.

The curvature of the contact discontinuity in the first approximation is given by

$$\lambda_{31} = {}^1/_2 (U_{1l} + C_l U_{2l})$$

3. Let us establish the forms of the functions K_0, L_0, Ω_0 occurring in the first-approximation formulas (1.11) of the general solution for the rarefaction wave in a medium with a binomial equation of state

$$e = \frac{p + \gamma p_*}{(\gamma - 1)\rho} - \frac{c_*^2}{\gamma - 1}$$

where $p_* = \gamma^{-1}\rho_* c_*^2$, γ, ρ_*, c_* are some constants. An ideal gas is a special case with $(c_* = 0)$.

Introducing the entropy s (more exactly, some function of the entropy), we can represent the density ρ, the velocity of sound a, and the specific internal energy e as functions of s and the pressure p:

$$e(p, s) = \frac{p + \gamma p_*}{(\gamma - 1)s}(p + p_*)^{-1/\gamma} - \frac{c_*^2}{\gamma - 1} \qquad (3.1)$$
$$\rho(p, s) = s(p + p_*)^{1/\gamma}, \quad a(p, s) = \sqrt{\gamma (p + p_*)^{1-1/\gamma}/s}$$

The formulas defining the solution in the region of the centred rarefaction wave for a medium with a binomial equation of state have the form

$$a(\lambda) = \frac{\gamma - 1}{\gamma + 1}\left[\pm(u_1 - \lambda) + \frac{2}{\gamma - 1} a_1\right] \qquad (3.2)$$
$$u(\lambda) = \pm a(\lambda) + \lambda,\ p(\lambda) = (p_1 + p_*)[a(\lambda)/a_1]^{2\gamma/(\gamma-1)} - p_*$$

where u_1, p_1 and a_1 are the parameters of the background against which the wave propagates; the upper sign corresponds to a wave propagating to the left and the lower sign to a wave propagating to the right.

Substituting (3.1) and (3.2) into (1.11), we obtain

$$K_0 = \pm\frac{1}{\rho a}\frac{3\gamma - 1}{\gamma + 5}, \quad L_0 = \mp\frac{2}{\gamma + 5}\frac{a}{s_1}$$
$$\Omega_0 = \pm\frac{8(\gamma + 1)}{(\gamma + 5)(3\gamma - 1)}\frac{a_1\sqrt{\rho_1 a_1}}{s_1}\left(\frac{a}{a_1}\right)^{(3\gamma-1)/(2\gamma-2)}$$

4. Using the relationships obtained above, let us consider the variation of the velocity of a steady shock wave moving through a gas at rest with the parameters ρ_{10}, p_{10} if, starting at some point $(x = 0)$, the density ahead of the wave starts varying in a manner which locally has the form

$$\rho_{10}(x) = \rho_{10} + R_{10}x + O(x^2)$$

This is obviously a special case of our problem, with a weak discontinuity on the left of the contact discontinuity and a shock wave on the right of the contact discontinuity. The constant C_r in (2.15), which in this case determines the variation of the shock wave velocity, is given by

$$C_r = -(c_{30}U_1 + P_1)/(c_{30}U_2 + P_2)$$

Substituting U_1, U_2, P_1 and P_2 from (2.6) into this ratio, we obtain, after some reduction, that the acceleration acquired by the shock wave on passing through the point $x = 0$ can be represented in the form

$$a_{sh} = -F\frac{a_{10}^2 R_{10}}{\rho_{10}}, \quad F = \frac{M_{30}(2\gamma M_{10}^2 + 1 - \gamma) - (\gamma - 1)(M_{10}^2 - 1)]}{M_{30}(2\gamma M_{10}^2 + 1 - \gamma) + 2M_{10}^2 + \gamma - 1}$$

where M_{10} is the Mach number of the shock wave, M_{30} is the Mach number behind the wave front. We can show that $F > 0$ for $M_{10} > 1$. Therefore the sign of a_{sh} is minus the sign of R_{10}, i.e., if the density ahead of the wave increases, the shock wave decelerates, and conversely. The absolute value of the acceleration increases as the square of M_{10}.

5. In conclusion we note the following. The problem of the breakup of an ordinary discontinuity plays an important role in numerical methods of the mechanics of continuous media. In particular, its solution is used to construct finite-difference schemes for the numerical integration of the unsteady equations of gas dynamics (so-called Godunov-type schemes /3/). The result is a numerical scheme of first-order approximation, which leads to certain errors in numerical calculations. The analytical solution of the generalized Riemann problem obtained in this paper may be used to improve the order of approximation of the Godunov scheme if the piecewise-constant approximation is replaced by a piecewise-linear approximation.

REFERENCES

1. BEN-ARTZI M. and FALKOVITZ J., A second-order Godunov-type scheme for compressible fluid dynamics, J. Comput. Phys., 55, 1, 1984.
2. KOCHIN N.E., On the theory of discontinuities in a fluid, in: Collected Papers, 2, Izd. Akad. Nauk SSSR, Moscow-Leningrad, 1949.
3. GODUNOV S.K., ZABRODIN A.V., IVANOV M.YA., et al., The Numerical Solution of Multidimensional Problems of Gas Dynamics, Nauka, Moscow, 1976.

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ON THE POSSIBLE MODES OF FLOW ROUND TAPERED BODIES OF FINITE THICKNESS AT ARBITRARY SUPERSONIC VELOCITIES OF THE APPROACH STREAM*

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Supersonic planar flow round a symmetric tapered body is considered for which, at each point, the angle of inclination of the wall is less than the limiting angle for the shock polar corresponding to the approach stream. It is shown that states of flow with the formation of both an attached shock wave (SW) of the strong family and a detached SW with subsequent subsonic flow between the shock wave, the body and the sonic line are impossible at any stream velocities. In essence, the results obtained by Nikol'skii /1/ are transferred to the case of an arbitrary Mach number of the approach stream.

The impossibility of flow round a finite wedge with the formation of an attached SW of the strong family has been proved when substantial simplifying assumptions are made in /2/. The problem has been considered in /1/ in a general formulation under the sole assumption that there are no local supersonic zones and closed stream lines in the subsonic flow domain between the SW, the body and the sonic line.

In this paper, the proof is based on a monotonic change in the angle of inclination of the velocity vector along lines of constant pressure (isobars). This fact, which is valid in the case of vortex flows, has been established previously in /1/ and the analogous result for

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